Mean-variance Hedging in the Discontinuous Case

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Abstract

The results on the mean-variance hedging problem in Gouriéroux, Laurent and Pham (1998), Rheinländer and Schweizer (1997) and Arai (2005) are extended to discontinuous semimartingale models. When the numéraire method is used, we only assume the Radon-Nikodym derivative of the variance-optimal signed martingale measure (VSMM) is non-zero almost surely (but may be strictly negative). When discussing the relation between the solutions and the Galtchouk-Kunita-Watanabe decompositions under the VSMM, we only assume the VSMM is equivalent to the reference probability.

JEL Classification: G10

2000 Mathematics Subject Classification: 91B28, 60H05, 60G48

Key words and phrases: Mean-variance hedging, variance-optimal martingale measure, numéraire, Galtchouk-Kunita-Watanabe decomposition

*Supported by the National Natural Science Foundation of China under grant 10571167. Email: xia@amss.ac.cn.

1 Introduction

Let S be a semimartingale and Θ a family of some S-integrable predictable processes ϑ such that the stochastic integral $G_T(\vartheta) := \int_0^T \vartheta_t \, dS_t \in L^2(\mathbb{P})$, where T is a positive time horizon, then $G_T(\Theta) := \{G_T(\vartheta) : \vartheta \in \Theta\}$ is a subspace of $L^2(\mathbb{P})$. The problem of mean-variance hedging is to approximate any contingent claim, i.e., any random variable $H \in L^2(\mathbb{P})$ by the elements in $G_T(\Theta)$. In order to guarantee the existence of the solution of such a problem, the working space Θ of admissible strategies should be chosen such that $G_T(\Theta)$ is closed in $L^2(\mathbb{P})$.

In the existing literature, the space Θ usually consists of all S-integrable predictable processes ϑ such that the stochastic integral $G(\vartheta) := \int \vartheta \, dS$ is a square-integrable semimartingale. If S is a (local) martingale, the closedness holds true by the definition of stochastic integration. If S is only a semimartingale, additional assumptions must be imposed to ensure the closedness. For a continuous semimartinagle, Delbaen et al. (1997) established necessary and sufficient conditions for the closedness. For further results along this line, see Grandits and Krawczyk (1998) and Choulli et al. (1998, 1999). When the problem of mean-variance hedging is studied, $G_T(\Theta)$ is usually assumed to be closed in $L^2(\mathbb{P})$ explicitly or implicitly under additional conditions, see Schweizer (1996), Rheinländer and Schweizer (1997) (RS 1997, for short), Hou and Karatzas (2004) and Arai (2005), among others. But all these additional conditions imposed on S are rather strong.

On the other hand, Delbaen and Schachermayer (1996b) defined the working space starting from "simple" strategies and discussed the duality relation between attainable claims (by admissible strategies) and equivalent martingale measures. The space chosen by them automatically has the $L^2(\mathbb{P})$-closedness. Inspired by this, for continuous semimartingale models, Gouriéroux, Laurent and Pham (1998) (GLP 1998, for short) dealt with the mean-variance hedging problem, using the same working space as in Delbaen and Schachermayer (1996b).

It is well known that the variance-optimal signed martingale measure (VSMM,

for short) plays an important role in studying the mean-variance hedging problem. For a continuous semimartingale model, it turns out that VSMM is equivalent to the reference probability measure, see Delbaen and Schachermayer (1996a). But in general, the VSMM is only a signed measure and therefore the set of equivalent martingale measures is not enough. Inspired by this and by a similar way of Delbaen and Schachermayer (1996b), when discussing Markowitz's portfolio selection problem, Xia and Yan (2006) defined a space of admissible strategies which has the $L^2(\mathbb{P})$-closedness and has the duality relation to signed martingale measures (rather than only to equivalent ones). In an independent work, Černý and Kallsen (2005) also observed this fact and chose the same space.

The aim of this paper is to investigate the mean-variance hedging problem for discontinuous models within the working space of Xia and Yan (2006). The results of RS 1997, GLP 1998 and Arai (2005) are extent to our settings. RS 1997 and GLP 1998 dealt with the continuous semimartingale model. Using a change of numéraire and a change of measure, GLP 1998 reduced the problem to a martingale framework. RS 1997 used the Galtchouk-Kunita-Watanabe decomposition (GKW decomposition, for short) under the VSMM and obtained a solution of feedback form. They also discussed the relation between their solutions and those of GLP 1998. Arai (2005) extended the results of GLP 1998 and RS 1997 to the discontinuous case under some additional assumptions on the VSMM: the VSMM is equivalent to the reference probability, the corresponding density process Z satisfies the reverse Hölder inequality and another condition on the jump of Z. But in our paper, when the numéraire method is used, we only assume the Radon-Nikodym derivative of the VSMM is non-zero almost surely (but may be strictly negative). When discussing the relation between the solutions and the GKW decompositions under the VSMM, we only assume the VSMM is equivalent to the reference probability. The other rather restrictive conditions such as the reverse Hölder inequality are removed here.

2 The market model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, where $\mathcal{F}_0 = \sigma\{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$, and T is a positive time horizon. Throughout this paper, $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is abbreviated as $L^2(\mathbb{P})$. For any $a, b \in \mathbb{R}$, we denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. All vectors are column vectors and the transposition of a vector is denoted by x^{tr}. For any $x, y \in \mathbb{R}^d$, the inner product of x and y is $x^{\mathrm{tr}}y$ and the Euclidean norm of x is $|x| := \sqrt{x^{\mathrm{tr}}x}$.

2.1 Simple strategies and signed martingale measures

In this subsection, we first introduce the definitions of simple strategies and signed martingale measures and then present some existing results.

Definition 2.1 *The family $\mathscr{L}^2(\mathbb{P})$ consists of all $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-progressively measurable processes S such that $\{S_U : U \text{ stopping time}\} \subset L^2(\mathbb{P})$. The family $\mathscr{L}^2_{\mathrm{loc}}(\mathbb{P})$ consists of all $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-progressively measurable processes S such that there exists a sequence $(U_n)_{n\ge1}$ of localizing stopping times increasing to T such that, for each $n \ge 1$, the stopped process $S^{U_n} \in \mathscr{L}^2(\mathbb{P})$.*

We make the following standing hypothesis on an $\mathbb{R}^d$-value process S, which models the (discounted) price processes of the risky assets:

(H0) $S \in \mathscr{L}^2_{\mathrm{loc}}(\mathbb{P})$.

Remark 2.1 Under (H0), S is not necessarily a semimartingale. We in fact don't even assume S is optional at the moment.

Definition 2.2 *We say a process ϑ is a simple trading strategy if ϑ has a form*

$$\vartheta = \sum_{i=1}^{n} h_i \mathbf{1}_{]\!]T_{1i},T_{2i}]\!]}, \tag{2.1}$$

where, for each $i = 1, \ldots, n$, $T_{1i} \le T_{2i}$ are stopping times such that $S^{T_{2i}} \in \mathscr{L}^2_{\mathrm{loc}}(\mathbb{P})$ and h_i is bounded $\mathbb{R}^d$-valued $\mathcal{F}_{T_{1i}}$-measurable. The space Θ^s consists of all simple trading strategies. For any $\vartheta \in \Theta^s$ having form (2.1)*, the stochastic integral of ϑ with respect to S is*

$$G_t(\vartheta) := (\vartheta \bullet S)_t = \sum_{i=1}^{n} h_i^{\mathrm{tr}}(S_{T_{2i}\wedge t} - S_{T_{1i}\wedge t}).$$

Obviously, $G_T(\Theta^s) := \{G_T(\vartheta) : \vartheta \in \Theta^s\}$ is a subspace in $L^2(\mathbb{P})$.

We will use the following notations:

- $\mathcal{D}^s := \{g \in L^2(\mathbb{P}) : \mathbb{E}[gf] = 0$ for all $f \in G_T(\Theta^s)$ and $\mathbb{E}[g] = 1\}$;
- $\mathcal{D}^e := \{g \in \mathcal{D}^s : g > 0$ a.s.$\}$;
- For any $g \in \mathcal{D}^s$, $\mathbb{Q}^g$ is the signed measure on $(\Omega, \mathcal{F})$ defined by $\frac{d\mathbb{Q}^g}{d\mathbb{P}} = g$;
- For any $g \in \mathcal{D}^s$, Z^g is the RCLL version of the martingale $(\mathbb{E}[g|\mathcal{F}_t])$;
- $\mathcal{M}^s := \{\mathbb{Q}^g : g \in \mathcal{D}^s\}$ and $\mathcal{M}^e = \{\mathbb{Q}^g : g \in \mathcal{D}^e\}$.

It is clear that $\mathcal{D}^s$ is convex and closed in $L^2(\mathbb{P})$. For each $g \in \mathcal{D}^s$ and each $i = 1, \ldots, d$, $S^i Z^g$ is a local martingale.

Definition 2.3 *Any element in $\mathcal{M}^s$ (resp. $\mathcal{M}^e$) is called a signed (resp. equivalent) martingale measure for S.*

Throughout this paper, the closure $\overline{\{\cdots\}}$ refers to the $L^2(\mathbb{P})$-norm. Then we have the following easy lemma, see, e.g., Lemma 2.1 of Delbaen and Schachermayer (1996a).

Lemma 2.1 *Under assumption (H0), we have:*

(a) $\mathcal{M}^s \neq \emptyset \Longleftrightarrow 1 \notin \overline{G_T(\Theta^s)}$;

(b) *For any $g \in L^2(\mathbb{P})$,*

$$g \in \mathcal{D}^s \iff \mathbb{E}[g] = 1 \text{ and } \mathbb{E}[gf] = 0 \text{ for all } f \in \overline{G_T(\Theta^s)}.$$

Lemma 2.1(b) and the following lemma, which goes back to Lemma 2.2 of Xia and Yan (2006), give the bipolar relation between $\overline{G_T(\Theta^s)}$ and $\mathcal{D}^s$.

Lemma 2.2 *Assume (H0) and $\mathcal{M}^s \neq \emptyset$, then we have:*

$$f \in \overline{G_T(\Theta^s)} \iff f \in L^2(\mathbb{P}) \text{ and } \mathbb{E}[fg] = 0 \text{ for all } g \in \mathcal{D}^s. \tag{2.2}$$

2.2 Admissible trading strategies

Subsequently, we always assume S satisfies the following condition:

(H1) S is an $\mathbb{R}^d$-valued RCLL semimartingale and $S \in \mathscr{L}^2_{\mathrm{loc}}(\mathbb{P})$.

The stochastic integral of a predictable process ϑ with respect to a semimartingale X is denoted as $\int \vartheta\, dX$ or $\vartheta \bullet X$. We denote by $\mathscr{L}(X)$ the set of all X-integrable predictable processes. For the theory of stochastic integration we refer to Jacod (1979), and Jacod and Shiryaev (1987); particularly, for vector stochastic integrals, see Jacod (1980), and Shiryaev and Cherny (2002). Following Xia and Yan (2006) (see also an independent work of Černý and Kallsen (2005)), we give the definition of admissible strategies below.

Definition 2.4 *An admissible trading strategy is a process $\vartheta \in \mathscr{L}(S)$ such that $G_T(\vartheta) := (\vartheta \bullet S)_T \in \overline{G_T(\Theta^s)}$. The space Θ consists of all admissible trading strategies and $G_T(\Theta) := \{G_T(\vartheta) :\ \vartheta \in \Theta\}$.*

Definition 2.5 *For any $\vartheta \in \mathscr{L}(S)$, (ϑ^j) is a sequence of simple strategies approximating to ϑ, if $(\vartheta^j) \subset \Theta^s$ and $G_T(\vartheta^j) \to G_T(\vartheta)$ in $L^2(\mathbb{P})$.*

By definitions, for any $\vartheta \in \mathscr{L}(S)$, ϑ is admissible if and only if it allows an approximating sequence of simple strategies. Lemma 2.2 yields

Lemma 2.3 *Under assumptions (H1) and that $\mathcal{M}^s \neq \emptyset$, for any $\vartheta \in \mathscr{L}(S)$, $\vartheta \in \Theta$ if and only if ϑ satisfies the following condition:*

$$\begin{cases} G_T(\vartheta) := \int_0^T \vartheta_t \, dS_t \in L^2(\mathbb{P}) \\ \\ \mathbb{E}[G_T(\vartheta)g] = 0 \quad \textit{for all } g \in \mathcal{D}^s. \end{cases} \tag{2.3}$$

Obviously, $\Theta^s \subset \Theta$, $G_T(\Theta^s) \subset G_T(\Theta)$ and $G_T(\Theta) \subset \overline{G_T(\Theta^s)}$. Thus we have $\overline{G_T(\Theta^s)} = \overline{G_T(\Theta)}$. Furthermore, the following theorem, which goes back to Xia and Yan (2006, Theorem 2.1 and Remark 2.3), shows that $G_T(\Theta)$ is automatically closed in $L^2(\mathbb{P})$, if we assume in addition that

(H2) $\mathcal{M}^e \neq \emptyset$.

Theorem 2.1 *Under assumptions (H1) and (H2), we have:*

(a) *For any $f \in \overline{G_T(\Theta^s)}$, there exists a $\vartheta \in \Theta$ such that $f = \int_0^T \vartheta_t \, dS_t$ and $\int \vartheta \, dS$ is a uniformly integrable $\mathbb{Q}$-martingale for each $\mathbb{Q} \in \mathcal{M}^e$;*

(b) $\overline{G_T(\Theta^s)} = G_T(\Theta)$.

Remark 2.2 Under assumptions of the previous theorem, if $\vartheta \in \Theta$ satisfies the conditions in (a) and $(\vartheta^j) \subset \Theta^s$ is an approximating sequence, then for any $\mathbb{Q} \in \mathcal{M}^e$, $(\vartheta^j \bullet S)_T \to (\vartheta \bullet S)_T$ in $L^1(\mathbb{Q})$. On the other hand, for any $\vartheta^j \in \Theta^s$, $\int \vartheta^j \, dS$ is a uniformly integrable $\mathbb{Q}$-martingale for each $\mathbb{Q} \in \mathcal{M}^e$ (see Lemma 2.4 below). Thus for any $\mathbb{Q} \in \mathcal{M}^e$ and any stopping time τ,

$$(\vartheta^j \bullet S)_\tau = \mathbb{E}_{\mathbb{Q}}[(\vartheta^j \bullet S)_T | \mathcal{F}_\tau] \xrightarrow{L^1(\mathbb{Q})} \mathbb{E}_{\mathbb{Q}}[(\vartheta \bullet S)_T | \mathcal{F}_\tau] = (\vartheta \bullet S)_\tau,$$

which implies

$$(\vartheta^j \bullet S)_\tau \xrightarrow{\mathbb{P}} (\vartheta \bullet S)_\tau \quad \text{for any stopping time } \tau. \tag{2.4}$$

This fact will be used in proving Theorem 2.2.

Remark 2.3 Kreps-Yan theorem (see, e.g., Schachermayer 2005) yields

$$\mathcal{M}^e \neq \emptyset \Longleftrightarrow \overline{G_T(\Theta^s) - L^2_+(\mathbb{P})} \bigcap L^2_+(\mathbb{P}) = \{0\}$$

Definition 2.6 *The space Θ^u consists of all processes $\vartheta \in \mathscr{L}(S)$ satisfying*

$$\begin{cases} G_T(\vartheta) := \int_0^T \vartheta_t \, dS_t \in L^2(\mathbb{P}) \\ (\vartheta \bullet S) Z^g \text{ is a uniformly integrable martingale for each } g \in \mathcal{D}^s. \end{cases} \tag{2.5}$$

Theorem 2.2 *Under assumptions (H1) and (H2), $G_T(\Theta) = G_T(\Theta^u)$.*

Proof. "$\supset$" is clear. Conversely, for any $f \in G_T(\Theta)$, by Theorem 2.1, there exists a $\vartheta \in \Theta$ such that $f = (\vartheta \bullet S)_T$ and $\vartheta \bullet S$ is a uniformly integrable $\mathbb{Q}$-martingale for each $\mathbb{Q} \in \mathcal{M}^e$. We shall show ϑ satisfies (2.5). The first line of (2.5) is obvious.

By Theorem 2.1, there exists an approximating sequence $(\vartheta^j) \subset \Theta^s$ for ϑ such that $(\vartheta^j \bullet S)_T \to (\vartheta \bullet S)_T$ in $L^2(\mathbb{P})$, whence for any $g \in \mathcal{D}^s$, $(\vartheta^j \bullet S)_T Z^g_T \to (\vartheta \bullet S)_T Z^g_T$ in $L^1(\mathbb{P})$. Moreover, Lemma 2.4 below yields that, for any $g \in \mathcal{D}^s$,

$$(\vartheta^j \bullet S)_t Z^g_t = \mathbb{E}[(\vartheta^j \bullet S)_T Z^g_T | \mathcal{F}_t] \xrightarrow{L^1(\mathbb{P})} \mathbb{E}[(\vartheta \bullet S)_T Z^g_T | \mathcal{F}_t]. \tag{2.6}$$

Then (2.6) and (2.4) imply $\mathbb{E}[(\vartheta \bullet S)_T Z^g_T | \mathcal{F}_t] = (\vartheta \bullet S)_t Z^g_t$ for each $g \in \mathcal{D}^s$ and therefore ϑ satisfies the second line of (2.5). $\square$

The following lemma has been used to prove Theorem 2.2.

Lemma 2.4 *Under assumptions (H1) and (H2), $\Theta^s \subset \Theta^u$.*

Proof. Let $\vartheta \in \Theta^s$. The first line of (2.5) is obvious. Let $\vartheta \in \Theta^s$ have form (2.1). For any $0 \leq s \leq t \leq T$ and $A \in \mathcal{F}_s$, we can see

$$\widetilde{\vartheta} := (\mathbf{1}_A \mathbf{1}_{\rrbracket s,t \rrbracket}) \vartheta = \sum_{i=1}^n \widetilde{h}_i \mathbf{1}_{\rrbracket \widetilde{T}_{1i}, \widetilde{T}_{2i} \rrbracket} \in \Theta^s,$$

where

$$\widetilde{T}_{1i} = (T_{1i} \vee s) \wedge (T_{2i} \wedge t), \quad \widetilde{T}_{2i} = T_{2i} \wedge t, \quad \widetilde{h}_i = h_i \mathbf{1}_A \mathbf{1}_{[(T_{1i} \vee s) \leq \widetilde{T}_{2i}]}.$$

Obviously,

$$(\widetilde{\vartheta} \bullet S)_T = (\mathbf{1}_A \mathbf{1}_{]\!]s,t]\!]}) \bullet (\vartheta \bullet S) = ((\vartheta \bullet S)_t - (\vartheta \bullet S)_s)\mathbf{1}_A.$$

For any $g \in \mathcal{D}^s$, we have

$$\begin{aligned}
\mathbb{E}[(\vartheta \bullet S)_t Z_t^g \mathbf{1}_A] - \mathbb{E}[(\vartheta \bullet S)_s Z_s^g \mathbf{1}_A] &= \mathbb{E}[(\vartheta \bullet S)_t Z_T^g \mathbf{1}_A] - \mathbb{E}(\vartheta \bullet S)_s Z_T^g \mathbf{1}_A] \\
&= \mathbb{E}[(\widetilde{\vartheta} \bullet S)_T g] \\
&= 0,
\end{aligned}$$

which implies $\mathbb{E}[(\vartheta \bullet S)_t Z_t^g | \mathcal{F}_s] = (\vartheta \bullet S)_s Z_s^g$ a.s. and therefore $(\vartheta \bullet S) Z^g$ is a uniformly integrable martingale. That is just the second line of (2.5). □

3 Variance-optimal signed martingale measure

It is easy to see that, under hypothesis (H1) and that $\mathcal{M}^s \neq \emptyset$, $\mathcal{D}^s$ is a non-empty closed convex subset of $L^2(\mathbb{P})$. Therefore there exists a unique $\mathbb{Q}^* \in \mathcal{M}^s$ such that $g^* := \frac{d\mathbb{Q}^*}{d\mathbb{P}}$ has minimal $L^2(\mathbb{P})$-norm in $\mathcal{D}^s$. The signed measure $\mathbb{Q}^*$ is called the variance-optimal signed martingale measure (VSMM, for short) for S, since g^* minimizes $\mathrm{Var}[g] = \mathbb{E}[g^2] - 1$ over $g \in \mathcal{D}^s$. By Lemma 2.1 of Delbaen and Schachermayer (1996a) and Lemma 3.2 below, we have:

Lemma 3.1 *If $\mathcal{M}^s \neq \emptyset$, then g^* is the unique element of $\overline{G_T(\Theta^s)} + \mathbb{R}$ such that (as a linear functional on $L^2(\mathbb{P})$) vanishing on $\overline{G_T(\Theta^s)}$ and equaling 1 on the constant function 1.*

Lemma 3.2 *If $\mathcal{M}^s \neq \emptyset$, then $\overline{G_T(\Theta^s) + \mathbb{R}} = \overline{G_T(\Theta^s)} + \mathbb{R}$.*

Proof. The "$\supset$" part is clear. Let $f \in \overline{G_T(\Theta^s) + \mathbb{R}}$, then there exist sequences $(f^j) \subset G_T(\Theta^s)$ and $(\delta^j) \subset \mathbb{R}$ such that $f^j + \delta^j \to f$ in $L^2(\mathbb{P})$. For any $g \in \mathcal{D}^s$, we have $\delta^j = \mathbb{E}[(f^j + \delta^j)g] \to \mathbb{E}[fg]$ and therefore $f^j \to f - \mathbb{E}[fg]$ in $L^2(\mathbb{P})$, which yields $f \in \overline{G_T(\Theta^s)} + \mathbb{R}$. Thus "$\subset$" part also holds. □

Under assumptions (H1) and (H2), by Lemma 3.1 and Theorems 2.1–2.2, there exist $\vartheta^* \in \Theta^u$ and $a \in \mathbb{R}$ such that $g^* = a + (\vartheta^* \bullet S)_T$. Then we have

$$\mathbb{E}[(g^*)^2] = \mathbb{E}[(a + (\vartheta^* \bullet S)_T)g^*] = a$$

and therefore

$$g^* = \mathbb{E}[(g^*)^2] + (\vartheta^* \bullet S)_T. \tag{3.1}$$

For any fixed $\mathbb{Q} \in \mathcal{M}^e$, let $\widetilde{Z}^*$ be the RCLL version of the $\mathbb{Q}$-martingale defined by

$$\widetilde{Z}^*_t = \mathbb{E}_{\mathbb{Q}}[g^*|\mathcal{F}_t], \quad t \in [0, T], \tag{3.2}$$

then by (3.1) and $\vartheta^* \in \Theta^u$, we have

$$\widetilde{Z}^*_t = \mathbb{E}[(g^*)^2] + (\vartheta^* \bullet S)_t, \quad t \in [0, T]. \tag{3.3}$$

Thus the definition of $\widetilde{Z}^*$ is independent of the choice of $\mathbb{Q} \in \mathcal{M}^e$. Moreover, for each $g \in \mathcal{D}^s$, $\widetilde{Z}^* Z^g$ is a uniformly integrable martingale since $\vartheta^* \in \Theta^u$. The above arguments lead to the following lemma, which extends Lemma 2.2 of Delbaen and Schachermayer (1996a).

Lemma 3.3 *Under assumptions (H1) and (H2), we have:*

(a) $\widetilde{Z}^*$*, as defined in* (3.2)*, is independent of the choice of* $\mathbb{Q} \in \mathcal{M}^e$*;*

(b) *There exists* $\vartheta^* := (\vartheta^{*1}, \ldots, \vartheta^{*d})^{\mathrm{tr}} \in \Theta^u$ *such that* (3.3) *holds;*

(c) *For each* $g \in \mathcal{D}^s$, $\widetilde{Z}^* Z^g$ *is a uniformly integrable martingale. In particular,* $\widetilde{Z}^* Z^*$ *is a uniformly integrable martingale, where* Z^* *is the RCLL version of the martingale defined by* $Z^*_t = \mathbb{E}[g^*|\mathcal{F}_t]$.

Lemma 3.4 *Under assumptions (H1) and (H2), we have:*

(a) $\widetilde{Z}^* Z^* \geq 0$*;*

(b) *Let τ be the first time when $\widetilde{Z}^* Z^*$ hits 0, that is,*

$$\tau = \inf\{t \in [0,T] : \widetilde{Z}^*_t Z^*_t = 0\} \text{ with } \inf \emptyset = \infty,$$

then $\widetilde{Z}^* Z^* \mathbf{1}_{[\![\tau,T]\!]} = 0$;

(c) *If $\mathbb{P}(g^* = 0) = 0$, then $\widetilde{Z}^* Z^* > 0$.*

Proof. By Lemma 3.3(c), we have for any $t \in [0,T]$ that $\widetilde{Z}^*_t Z^*_t = \mathbb{E}[(g^*)^2 | \mathcal{F}_t] \geq 0$ a.s., and therefore (a) holds since $\widetilde{Z}^* Z^*$ is RCLL. Obviously, (a) and (b) imply (c). It remains to show (b).

For any stopping time σ with $0 \leq \sigma \leq T$ a.s., Doob stopping theorem leads to

$$\mathbb{E}[\widetilde{Z}^*_\sigma Z^*_\sigma] = \mathbb{E}[\widetilde{Z}^*_{\sigma\wedge\tau} Z^*_{\sigma\wedge\tau}] = \mathbb{E}[\widetilde{Z}^*_\sigma Z^*_\sigma \mathbf{1}_{[\sigma<\tau]}]$$

and hence $\mathbb{E}[X_\sigma] = \mathbb{E}[\widetilde{Z}^*_\sigma Z^*_\sigma \mathbf{1}_{[\sigma\geq\tau]}] = 0$, where $X = \widetilde{Z}^* Z^* \mathbf{1}_{[\![\tau,T]\!]}$. Then by section theorem, $X = 0$ indistinguishably, that is just (b). □

4 Mean-variance hedging

In this section, we always assume (H1) and (H2). The mean-variance hedging problem is, for any $H \in L^2(\mathbb{P})$, to

$$\text{Minimize } \mathbb{E}[(H - G_T(\vartheta))^2] \quad \text{subject to } \vartheta \in \Theta. \tag{4.1}$$

4.1 $L^2(\mathbb{P})$-orthogonal decomposition

Recalling Theorem 2.1(b) and Lemma 3.2, we know $G_T(\Theta) = \overline{G_T(\Theta^s)}$ and $G_T(\Theta) + \mathbb{R} = \overline{G_T(\Theta^s) + \mathbb{R}}$. So both $G_T(\Theta)$ and $G_T(\Theta) + \mathbb{R}$ are closed subspaces of $L^2(\mathbb{P})$. Thus (4.1) always allows a solution. By Lemma 2.1(b), we know

$$\mathcal{D}^s - g^* := \{g - g^* : g \in \mathcal{D}^s\}$$

is a closed subspace of $L^2(\mathbb{P})$ and by Lemma 2.2, $G_T(\Theta)+\mathbb{R} = (\mathcal{D}^s - g^*)^\perp$. Hereafter, $\{\dots\}^\perp$ stands for the orthogonal complement in $L^2(\mathbb{P})$. We denote by π the projection in $L^2(\mathbb{P})$ on $G_T(\Theta)^\perp$, then $\pi(1) \in G_T(\Theta)+\mathbb{R}$ and by Lemma 3.1, $g^* = \frac{\pi(1)}{\mathbb{E}[\pi(1)]}$. The above arguments lead to an orthogonal decomposition of $G_T(\Theta)+\mathbb{R}$ as follows:

$$G_T(\Theta)+\mathbb{R} = G_T(\Theta) \oplus g^*\mathbb{R},$$

where $g^*\mathbb{R}$ is the linear space spanned by g^*, that is, $g^*\mathbb{R} = \{\alpha g^* : \alpha \in \mathbb{R}\}$. Moreover, $L^2(\mathbb{P})$ can be orthogonally decomposed as follows:

$$L^2(\mathbb{P}) = G_T(\Theta) \oplus g^*\mathbb{R} \oplus (\mathcal{D}^s - g^*).$$

Consequently, by Theorem 2.2, we have the following easy theorem.

Theorem 4.1 *Assume (H1) and (H2). Any $H \in L^2(\mathbb{P})$ admits a unique orthogonal decomposition*

$$H = G_T(\vartheta^H) \oplus \alpha^H g^* \oplus (g^H - g^*), \tag{4.2}$$

where $\vartheta^H := (\vartheta^{H1}, \dots, \vartheta^{Hd})^{\mathrm{tr}} \in \Theta^u$, $\alpha^H = \frac{\mathbb{E}[Hg^*]}{\mathbb{E}[(g^*)^2]}$ and $g^H \in \mathcal{D}^s$. Moreover, ϑ^H solves (4.1).

4.2 Numéraire approach

Hereafter, in addition to (H1) and (H2), we always assume

(H3) $\mathbb{P}(g^* = 0) = 0$.

By Lemma 3.4, $\widetilde{Z}^* Z^* > 0$.

Following GLP 1998, see also RS 1997 and Arai (2005), we define an $\mathbb{R}^{d+1}$-valued process Y and a new probability measure $\widetilde{\mathbb{P}}$ as follows:

$$\begin{aligned} Y^0 &:= (\widetilde{Z}^*)^{-1}, \\ Y^i &:= S^i(\widetilde{Z}^*)^{-1}, \quad i = 1, \dots, d, \\ \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} &:= \frac{(g^*)^2}{\mathbb{E}[(g^*)^2]} = \frac{(\widetilde{Z}^*_T)^2}{\mathbb{E}[(g^*)^2]}. \end{aligned}$$

It is clear that, for any $H \in L^2(\mathbb{P})$ and $\vartheta \in \Theta$,

$$\mathbb{E}[(H - G_T(\vartheta))^2] = \mathbb{E}[(g^*)^2] \cdot \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left(\frac{H}{\widetilde{Z}^*_T} - \frac{G_T(\vartheta)}{\widetilde{Z}^*_T}\right)^2\right]. \tag{4.3}$$

The following notations will be used:

- The space $\mathscr{M}(\widetilde{\mathbb{P}})$ (resp. $\mathscr{M}_{\mathrm{loc}}(\widetilde{\mathbb{P}})$) consists of all uniformly integrable (resp. local) $\widetilde{\mathbb{P}}$-martingales;
- The space $\mathscr{M}^2(\widetilde{\mathbb{P}})$ (resp. $\mathscr{M}^2_{\mathrm{loc}}(\widetilde{\mathbb{P}})$) consists of all (resp. locally) square-integrable $\widetilde{\mathbb{P}}$-martingales and $\mathscr{M}^2_0(\widetilde{\mathbb{P}}) = \{M \in \mathscr{M}^2(\widetilde{\mathbb{P}}) : M_0 = 0\}$;
- The space Ψ consists of all processes $\psi \in \mathscr{L}(Y)$ such that $\psi \bullet Y \in \mathscr{M}^2(\widetilde{\mathbb{P}})$;
- The space $\widetilde{\Theta}$ consists of all processes $\vartheta \in \mathscr{L}(S)$ satisfying $\int_0^T \vartheta_t \, dS_t \in L^2(\mathbb{P})$ and $(\vartheta \bullet S)Z^*$ is a uniformly integrable martingale.

Lemma 2.4 shows $\Theta^s \subset \Theta^u \subset \widetilde{\Theta}$.

Proposition 4.1 *Under assumptions (H1), (H2) and (H3), we have*

$$\frac{1}{\widetilde{Z}^*_T} G_T(\Theta^u) = \{(\psi \bullet Y)_T : \psi \in \Psi\}. \tag{4.4}$$

Moreover, the relation between $\vartheta \in \Theta^u$ and $\psi \in \Psi$ is given by

$$\begin{aligned} \psi^i &= \vartheta^i, \quad i = 1, \dots, d, \\ \psi^0 &= (\vartheta \bullet S) - \vartheta^{\mathrm{tr}} S \end{aligned}$$

and

$$\vartheta^i = \psi^i + \vartheta^{*i}(\psi \bullet Y - \psi^{\mathrm{tr}} Y), \quad i = 1, \dots, d, \tag{4.5}$$

where ϑ^* is defined as in Lemma 3.3(b).

Proof. The relation between ϑ and ψ and the fact

$$\{\vartheta \bullet S : \vartheta \in \mathscr{L}(S)\} = \{(\psi \bullet Y)\widetilde{Z}^* : \psi \in \mathscr{L}(Y)\} \tag{4.6}$$

has been proved in Proposition 8 of RS 1997.

In order to prove the "$\subset$" part of (4.4), it is enough to show the corresponding ψ is in Ψ if $\vartheta \in \widetilde{\Theta}$. Actually, let $\vartheta \in \widetilde{\Theta}$, then $(\vartheta \bullet S)_T \in L^2(\mathbb{P})$ and $(\vartheta \bullet S)Z^*$ is a uniformly integrable martingale. By (4.6), $\vartheta \bullet S = (\psi \bullet Y)\widetilde{Z}^*$ and therefore $(\psi \bullet Y)\widetilde{Z}^* Z^*$ is a uniformly integrable martingale. On the other hand, Lemma 3.3(c) shows

$$\mathbb{E}\left[\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right| \mathcal{F}_t\right] = \frac{\widetilde{Z}^*_t Z^*_t}{\mathbb{E}[(g^*)^2]}, \tag{4.7}$$

and therefore $(\psi \bullet Y) \in \mathscr{M}(\widetilde{\mathbb{P}})$. Moreover,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}[(\psi \bullet Y)_T^2] = \frac{\mathbb{E}[(\vartheta \bullet S)_T^2]}{\mathbb{E}[(g^*)^2]} < \infty.$$

Thus $\psi \in \Psi$, which implies the "$\subset$" part of (4.4).

It is worth noting that the argument in the previous paragraph yields particularly that $Y^i \in \mathscr{M}^2_{\mathrm{loc}}(\widetilde{\mathbb{P}})$ for each $i = 0, 1, \ldots, d$, since $S \in \mathscr{L}^2_{\mathrm{loc}}(\mathbb{P})$ and $\Theta^s \subset \widetilde{\Theta}$. On the other hand, by Lemma 3.3(c) , for each $g \in \mathcal{D}^s$, $Z^g\widetilde{Z}^*$ is a uniformly integrable martingale, and then by (4.7), $Z^g(Z^*)^{-1} \in \mathscr{M}(\widetilde{\mathbb{P}})$. Obviously,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}[(Z^g_T(Z^*_T)^{-1})^2] = \frac{\mathbb{E}[g^2]}{\mathbb{E}[(g^*)^2]} < \infty,$$

thus $Z^g(Z^*)^{-1} \in \mathscr{M}^2(\widetilde{\mathbb{P}})$. Moreover, by $\Theta^s \subset \Theta^u$ and by $S \in \mathscr{L}^2_{\mathrm{loc}}(\mathbb{P})$, we can see, for each $i = 1, \ldots, d$ and each $g \in \mathcal{D}^s$, $S^i Z^g$ is a local martingale, that is, $Y^i Z^g(Z^*)^{-1} \in \mathscr{M}_{\mathrm{loc}}(\widetilde{\mathbb{P}})$. Similarly, for each $g \in \mathcal{D}^s$, Z^g is a square-integrable martingale, that is, $Y^0 Z^g(Z^*)^{-1} \in \mathscr{M}^2(\widetilde{\mathbb{P}})$. To conclude this paragraph, we have $\langle Y^i, Z^g(Z^*)^{-1}\rangle(\widetilde{\mathbb{P}}) = 0$, for each $i = 0, 1, \ldots, d$ and each $g \in \mathcal{D}^s$.

Now we are in a position to prove the "$\supset$" part. Let $\psi \in \Psi$ and the corresponding ϑ be given by (4.5), we should show $\vartheta \in \Theta^u$. Actually, by the results in the previous

paragraph, we have for each $g \in \mathcal{D}^s$ that $\langle \psi \bullet Y, Z^g(Z^*)^{-1} \rangle(\widetilde{\mathbb{P}}) = 0$ and therefore $(\psi \bullet Y)Z^g(Z^*)^{-1} \in \mathscr{M}(\widetilde{\mathbb{P}})$. That is, by (4.6) , $(\vartheta \bullet S)(\widetilde{Z}^*)^{-1}Z^g(Z^*)^{-1} \in \mathscr{M}(\widetilde{\mathbb{P}})$, and then by (4.7), $(\vartheta \bullet S)Z^g$ is a uniformly integrable martingale, for each $g \in \mathcal{D}^s$. Now we have shown $\vartheta \in \Theta^u$. □

Actually, in the proof of the previous proposition, we have implicitly shown the following

Corollary 4.1 *Under assumptions (H1), (H2) and (H3), we have* $\widetilde{\Theta} = \Theta^u$.

Remark 4.1 For continuous semimartingale models, Θ^u was defined as the working space of admissible strategies in GLP 1998 and $\widetilde{\Theta}$ was used by RS 1997. In the continuous case, since $\mathbb{Q}^* \in \mathcal{M}^e$, the previous corollary shows $\Theta^u = \widetilde{\Theta}$. For the discontinuous case, when $\mathbb{Q}^* \in \mathcal{M}^e$, $\widetilde{\Theta}$ was used in Arai (2005). Anyway, we have $\Theta^u = \widetilde{\Theta}$ if (H3) is satisfied.

In view of Proposition 4.1 and (4.3), (4.1) is equivalent to the problem to

$$\text{Minimize } \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left(\frac{H}{\widetilde{Z}_T^*} - \int_0^T \psi \, dY\right)^2\right] \quad \text{over } \psi \in \Psi. \tag{4.8}$$

The solution of (4.8) is given by the GKW decomposition

$$\frac{H}{\widetilde{Z}_T^*} = \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\frac{H}{\widetilde{Z}_T^*}\right] + \int_0^T \psi_t^H \, dY_t + L_T^H, \tag{4.9}$$

under $\widetilde{\mathbb{P}}$, where $\psi^H := (\psi^{H0}, \psi^{H1}, \dots, \psi^{Hd})^{\mathrm{tr}} \in \Psi$ and $L^H \in \mathscr{M}_0^2(\widetilde{\mathbb{P}})$ is strongly $\widetilde{\mathbb{P}}$-orthogonal to Y.

The following theorem, which extends the corresponding results of GLP 1998, RS 1997 and Arai (2005) to our situation, shows the relation between decompositions (4.2) and (4.9).

Theorem 4.2 *Assume (H1), (H2) and (H3). For decompositions* (4.2) *and* (4.9), *we have*

$$\vartheta^{Hi} = \psi^{Hi} + \vartheta^{*i}(\psi^H \bullet Y - (\psi^H)^{\mathrm{tr}} Y), \quad i = 1, \dots, d, \tag{4.10}$$

$$g^H - g^* = L_T^H g^*. \tag{4.11}$$

Proof. Obviously, $\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\frac{H}{\widetilde{Z}^*_T}\right] = \frac{\mathbb{E}[Hg^*]}{\mathbb{E}[(g^*)^2]} = \alpha^H$. Proposition 4.1 and (4.3) imply (4.10), since ψ^H solves (4.8). By (4.9), we have

$$\begin{aligned} H &= \widetilde{Z}^*_T \int_0^T \psi^H_t \, dY_t + \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\frac{H}{\widetilde{Z}^*_T}\right] \widetilde{Z}^*_T + L^H_T \widetilde{Z}^*_T \\ &= G_T(\vartheta^H) + \alpha^H g^* + L^H_T g^*, \end{aligned}$$

which, by (4.2), implies (4.11). □

4.3 Decompositions under VSMM

In this subsection we study two kinds of decompositions under the VSMM $\mathbb{Q}^*$: one is directly derived from decompositions (4.2) and the other one is the GKW decomposition. Under assumption (H3), by Lemma 3.4, $\widetilde{Z}^* Z^* > 0$ indistinguishably. For any $H \in L^2(\mathbb{P})$, we can define V^H as follows:

$$V^H_t = \frac{\mathbb{E}[HZ^*_T|\mathcal{F}_t]}{Z^*_t} \quad \text{for all } t \in [0,T].$$

Obviously, $V^H Z^*$ is a uniformly integrable martingale. If $\mathbb{Q}^* \in \mathcal{M}^e$, then $V^H_t = \mathbb{E}_{\mathbb{Q}^*}[H|\mathcal{F}_t]$.

In the following theorem, we introduce a decomposition under $\mathbb{Q}^*$ which is directly derived from decompositions (4.2).

Theorem 4.3 *Assume (H1), (H2) and (H3). For any $H \in L^2(\mathbb{P})$, V^H has a unique decomposition*

$$V^H = V^H_0 + \varphi^H \bullet S + K^H, \tag{4.12}$$

where $V^H_0 = \mathbb{E}[Hg^*]$, $\varphi^H \in \Theta^u$, $K^H_0 = 0$, $K^H Z^*$ is a uniformly integrable martingale and $K^H_T \in (G_T(\Theta) + \mathbb{R})^\perp$. The relation between decompositions (4.2) *and* (4.12) *is:*

$$\begin{aligned} \varphi^H &= \vartheta^H + \alpha^H \vartheta^*, & (4.13) \\ K^H_t &= \frac{\mathbb{E}[(g^H - g^*)Z^*_T|\mathcal{F}_t]}{Z^*_t}. & (4.14) \end{aligned}$$

Moreover, we have

$$K^H = L^H \widetilde{Z}^*, \tag{4.15}$$

where L^H *is given by decomposition* (4.9).

Proof. "Uniqueness": In addition to (4.12), suppose V^H has another decomposition $V^H = V_0^H + \varphi \bullet S + K$ with φ and K satisfying the corresponding conditions of φ^H and K^H respectively, then $(\varphi - \varphi^H) \bullet S = K^H - K$ and therefore $K_T^H - K_T \in G_T(\Theta)$. On the other hand, the decompositions require $K_T^H - K_T \in (G_T(\Theta) + \mathbb{R})^\perp$. Thus $K_T^H = K^T$ a.s. Since both $K^H Z^*$ and KZ^* are uniformly integrable martingales, we have further that $K^H = K$ and hence $\varphi^H \bullet S = \varphi \bullet S$. This completes the uniqueness.

"Existence": For any $H \in L^2(\mathbb{P})$, H has decomposition (4.2). By $\vartheta^H \Theta^u$ and Lemma 3.3(c), we have

$$\mathbb{E}[HZ_T^* | \mathcal{F}_t] = G_t(\vartheta^H) Z_t^* + \alpha^H \widetilde{Z}_t^* Z_t^* + \mathbb{E}[(g^H - g^*) g^* | \mathcal{F}_t],$$

that is, by (3.3),

$$V_t^H = G_t(\vartheta^H) + \alpha^H \widetilde{Z}_t^* + K_t^H = \mathbb{E}[Hg^*] + G_t(\varphi^H) + K_t^H,$$

where φ^H and K^H are respectively given by (4.13) and (4.14). It is easy to verify that $\varphi^H \in \Theta^u$, $K_0^H = 0$, $K^H Z^*$ is a uniformly integrable martingale and $H_T^H = g^H - g^* \in (G_T(\Theta) + \mathbb{R})^\perp$.

Finally, by (4.7), $L^H \widetilde{Z}^* Z^*$ is a uniformly integrable martingale since $L^H \in \mathscr{M}_0^2(\widetilde{\mathbb{P}})$. Then (4.11) and (4.14) yield (4.15). □

Remark 4.2 In decomposition (4.12), since $K_T^H \in (G_T(\Theta) + \mathbb{R})^\perp$, (V_0^H, φ^H) solves the problem to

$$\text{Minimize } \mathbb{E}[(H - x - (\varphi \bullet S)_T)^2] \quad \text{over } (x, \varphi) \in \mathbb{R} \times \Theta.$$

Now we consider the GKW decompositions of local martingales under $\mathbb{Q}^*$. To this end, we assume further that $\mathbb{Q}^* \in \mathcal{M}^e$.

The GKW decomposition of V^H is:

$$V^H = V_0^H + \eta^H \bullet S + N^H, \tag{4.16}$$

where $V_0^H = \mathbb{E}[Hg^*]$, $\eta^H \in \mathscr{L}(S)$, and N^H satisfies $N_0^H = 0$ and both N^H and $N^H S$ are local $\mathbb{Q}^*$-martingales.

On the other hand, by Theorem 4.3, V^H can be decomposed as follows:

$$V^H = V_0^H + (\vartheta^H + \alpha^H \vartheta^*) \bullet S + L^H \widetilde{Z}^*.$$

Let

$$J^H := L^H \widetilde{Z}^* - L_-^H \bullet \widetilde{Z}^*, \tag{4.17}$$

then by (3.3), we have

$$V^H = V_0^H + (\vartheta^H + (\alpha^H + L_-^H)\vartheta^*) \bullet S + J^H, \tag{4.18}$$

Obviously, J^H is a local $\mathbb{Q}^*$-martingale with $J_0^H = 0$.

Assume the GKW decomposition of J^H is

$$J^H = \eta^J \bullet S + N^J, \tag{4.19}$$

where $\eta^J \in \mathscr{L}(S)$, $N_0^J = 0$, and both N^J and $N^J S$ are local $\mathbb{Q}^*$-martingales. In view of (4.16), (4.18) and (4.19), by the uniqueness of the GKW decomposition, we have $N^H = N^J$ and

$$\eta^H = \vartheta^H + (\alpha^H + L_-^H)\vartheta^* + \eta^J. \tag{4.20}$$

The above arguments lead to the following theorem, which extends Theorem 4.1 of Arai (2005) to our settings.

Theorem 4.4 *Assume (H1), (H2) and $\mathbb{Q}^* \in \mathcal{M}^e$. Let $H \in L^2(\mathbb{P})$, if V^H and J^H allow the GKW decompositions* (4.16) *and* (4.19) *respectively, then the solution ϑ^H of* (4.1) *satisfies the following feedback equation:*

$$\vartheta^H = \eta^H - \eta^J - \frac{\vartheta^*}{\widetilde{Z}^*_-}(V^H_- - G_-(\vartheta^H)). \tag{4.21}$$

Proof. By integration by parts and (3.3), we have

$$\begin{aligned}\widetilde{Z}^*(\alpha^H + L^H) &= \alpha^H \widetilde{Z}^*_0 + ((\alpha^H + L^H_-)\vartheta^*) \bullet S + \widetilde{Z}^*_- \bullet L^H + [\widetilde{Z}^*, L^H] \\ &= V^H - G(\vartheta^H) \qquad\qquad \text{(by (4.18))}\end{aligned}$$

and therefore

$$\alpha^H + L^H_- = \frac{V^H_- - G_-(\vartheta^H)}{\widetilde{Z}^*_-}.$$

Then by (4.20),

$$\vartheta^H = \eta^H - \eta^J - (\alpha^H + L^H_-)\vartheta^* = \eta^H - \eta^J - \frac{\vartheta^*}{\widetilde{Z}^*_-}(V^H_- - G_-(\vartheta^H)).$$

□

Proposition 4.2 *Under the conditions of Theorem 4.4, $\eta^J = 0$ if and only if $\sum \Delta L^H \Delta \widetilde{Z}^* \Delta S$ is a local $\mathbb{Q}^*$-martingale. If it is the case, then the solution ϑ^H of* (4.1) *satisfies the following feedback equation:*

$$\vartheta^H = \eta^H - \frac{\vartheta^*}{\widetilde{Z}^*_-}(V^H_- - G_-(\vartheta^H)), \tag{4.22}$$

where η^H is given by the GKW decomposition (4.16)*.*

Proof. Obviously, in decomposition (4.19), $\eta^J = 0$ if and only if $[J^H, S]$ is a local $\mathbb{Q}^*$-martingale. By integration by parts, one can compute that

$$[J^H, S] = [\widetilde{Z}^*_- \bullet L^H, S] + [[L^H, \widetilde{Z}^*], S] = \widetilde{Z}^*_- \bullet [L^H, S] + \sum \Delta L^H \Delta \widetilde{Z}^* \Delta S.$$

By Lemma 4.1 below, $\widetilde{Z}^*_- \bullet [L^H, S]$ is a local $\mathbb{Q}^*$-martingale, and therefore the conclusion of the proposition follows. □

Remark 4.3 If S is a continuous semimartingale satisfying (H1) and (H2), then $\mathbb{Q}^* \in \mathcal{M}^e$ automatically holds. In this case, due to the continuity of S, it always holds that $\sum \Delta L^H \Delta \widetilde{Z}^* \Delta S = 0$ and therefore Proposition 10 of RS 1997 can be recovered here.

Remark 4.4 In general, however, it is rather restrictive to assume $\sum \Delta L^H \Delta \widetilde{Z}^* \Delta S$ is a local $\mathbb{Q}^*$-martingale. This fact was observed by Arai (2005).

The following lemma has been used to prove Proposition 4.2.

Lemma 4.1 *Under the conditions of Theorem 4.4, for any H, $[L^H, S]$ is a local $\mathbb{Q}^*$-martingale.*

Proof. Since L^H is strongly $\widetilde{\mathbb{P}}$-orthogonal to Y, we know $L^H Y^i \in \mathscr{M}(\widetilde{\mathbb{P}})$ for each $i = 0, \dots, d$. For $i = 0$, $L^H Y^0 = L^H (\widetilde{Z}^*)^{-1}$ and therefore L^H is a uniformly integrable $\mathbb{Q}^*$-martingale. For each $i = 1, \dots, d$, $L^H Y^i = L^H S^i (\widetilde{Z}^*)^{-1}$ and therefore $L^H S^i$ is a uniformly integrable $\mathbb{Q}^*$-martingale. On the other hand, S is a $\mathbb{Q}^*$-martingale and therefore we can see $[L^H, S]$ is a local $\mathbb{Q}^*$-martingale. $\square$

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